Strategic games

Theory of Individual and Strategic Decisions

MSc Human Decision Science

Maastricht University

- **Game Theory**: strategic interaction among multiple players

Do you remember our discussion in Week 1 about strategic uncertainty and why the difference between one and two individuals is significant?

Example (Battle of the Sexes (BoS))

Ann and Bob each decide independently where they go, Basketball game (B) or Opera (O). Their preferences look as follows:

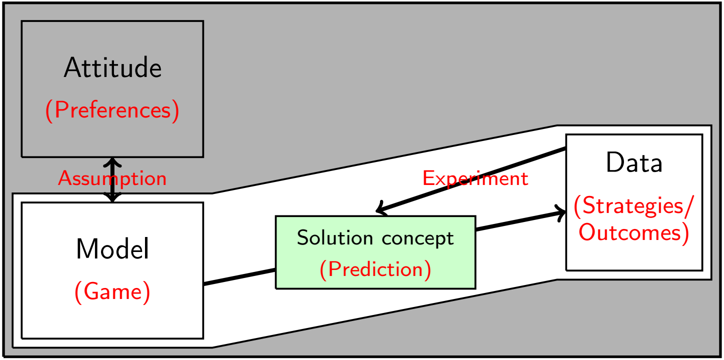
- **Ann:** Both Opera $\succ^a$ Both Basketball $\succ^a$ Not together
- **Bob:** Both Basketball $\succ^b$ Both Opera $\succ^b$ Not together

Ann's choice depends on her belief (subjective probability α of Bob going to B), i.e., each of her own choices becomes a lottery:

$$\begin{aligned} B &= (\alpha \times \text{Both Basketball}, (1-\alpha) \times \text{Not together}) \\ O &= (\alpha \times \text{Not together}, (1-\alpha) \times \text{Both Opera}) \end{aligned}$$

But she knows that the same applies to Bob. So, her beliefs are no longer exogenous: they are the product of strategic reasoning.

- **Decision Theory**:
  1. We represent (vNM) preferences with expected utility model
  2. We elicit Bernoulli utilities
- **Game Theory**:
  1. We exogenously assume that the individual has vNM preferences and that we already know the Bernoulli utilities (part of the game description)
  2. We use models of strategic reasoning to make predictions



Game description contains two parts:

1. The interaction translated in math (observable):
   - The players $N = \{1, \ldots, n\}$
   - The (available) actions/strategies A^i for each player i
   - One outcome per strategy profile $o(a^1, \ldots, a^n)$
2. The attitudes (unobservable, but assumed):
   - Preferences $\succeq^i$ represented by a Bernoulli utility u^i over the set of outcomes for each player i

Example (Battle of the Sexes (BoS))

- Players $N = \{\text{Ann } (a),\ \text{Bob } (b)\}$
- Strategies $A^i = \{\text{Basketball } (B),\ \text{Opera } (O)\}$
- Outcomes $Z = \{\underbrace{\text{Both Basketball}}_{o(B,B)},\ \underbrace{\text{Both Opera}}_{o(O,O)},\ \underbrace{\text{Not together}}_{o(O,B)=o(B,O)}\}$
- Ann's preferences:

$$u^a(\text{Both Basketball}) = 1,\ u^a(\text{Both Opera}) = 3,\ u^a(\text{Not together}) = 0$$

- Bob's preferences:

$$u^b(\text{Both Basketball}) = 3,\ u^b(\text{Both Opera}) = 1,\ u^b(\text{Not together}) = 0$$

- The game description is usually summarized on a table:

	B	O
B	1 , 3	0 , 0
O	0 , 0	3 , 1

Battle of the Sexes

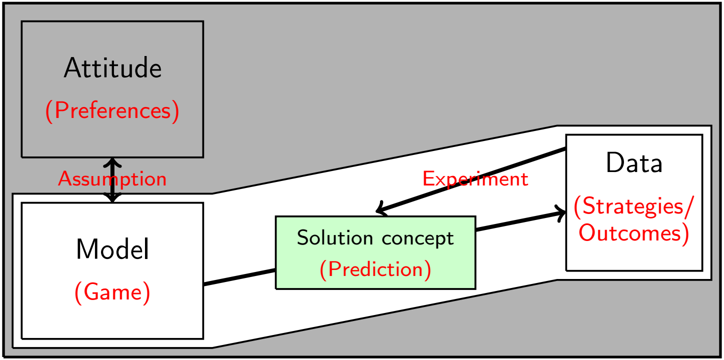
- Ann chooses the rows, Bob chooses the columns
- In each cell we write utilities of the two players

We do not write the outcomes explicitly

- The solution concept makes assumption about

  beliefs of players about the opponent's strategy

- We need this in order to formally describe strategic reasoning.
- Beliefs are unobservable
- In your textbook this is not explicitly introduced (for reasons I will explain in a bit)



- Look at the game from Ann's point of view:

	(α) B	$(1-\alpha)$ O
B	1 , 3	0 , 0
O	0 , 0	3 , 1

Battle of the Sexes

- Suppose her belief assigns probability α to Bob choosing B
- Then, her expected utility from each strategy is:

$$
\begin{aligned}
U^a(B,\alpha) &= \alpha \cdot 1 + (1-\alpha)\cdot 0 = \alpha \\
U^a(O,\alpha) &= \alpha \cdot 0 + (1-\alpha)\cdot 3 = 3(1-\alpha)
\end{aligned}
$$

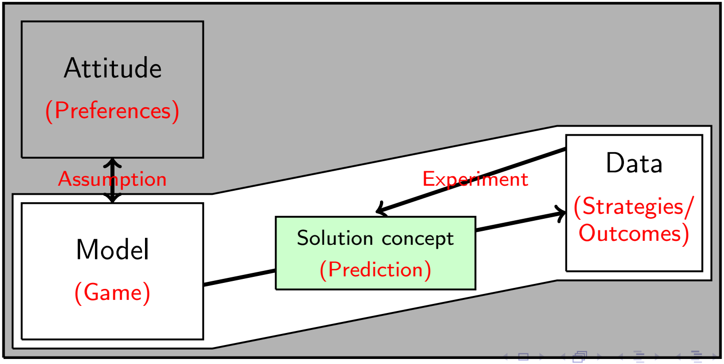
- So, $B \succeq^a O \Leftrightarrow \alpha \geq 3(1-\alpha) \Leftrightarrow \alpha \geq 3/4$
- Our prediction (about what Ann will do depends) on what she thinks Bob will do, i.e., on her strategic reasoning

How she reasons strategically is assumed by each solution concept

- **Solution concept:** a black box with
  - Input: the game description
  - Output: predictions in strategies (next week also in outcomes)
- Predictions are not necessarily unique
- There are different solution concepts:
  1. **Nash equilibrium** (today)
  2. **Iterated strict/weak dominance** (not in the course)
- Each solution concept is based on different

  assumptions about how players reason

  i.e., beliefs are used to make sense what is in the black box



- A **Nash Equilibrium (NE)** reflects the idea of stability

  A strategy profile from which no player unilaterally deviates

Definition

A strategy profile $(a^1, \ldots, a^n)$ is a NE if for every $i \in N$:

$$(a^i, a^{-i}) \succeq^i (x^i, a^{-i}) \text{ for every } x^i \in A^i$$

or equivalently

$$u^i(a^i, a^{-i}) \geq^i u^i(x^i, a^{-i}) \text{ for every } x^i \in A^i$$

- In a Nash Equilibrium every player i chooses optimally

  given the strategy a^{-i} that the opponents actually play
- Essentially we assume the following:
  1. **Behavior:** All players choose optimally given their beliefs
  2. **Strategic reasoning:** All players have correct beliefs
- This is the reason they omit beliefs in the book
- Other solution concepts do not assume correct beliefs
- Is it a reasonable assumption? We will come back to this.

Remark

There may exist multiple NE in a game.

Remark

A NE cannot involve strictly dominated strategies.

Remark

A NE may involve weakly dominated strategies.

Remark

There may exist no NE.

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Example (Coordination game)

Two subjects are left in NY and are asked to meet either at the Empire State Building (A) or at Grand Central Station (B). They cannot communicate in any way. If they manage to meet they will receive $100 each. Else, they will not get anything. The utilities look as follows:

	A	B
A	1 , 1	0 , 0
B	0 , 0	1 , 1

Which are the NE of this game?

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A NE cannot involve strictly dominated strategies.

Example (Prisoner's dilemma)

Two suspects are interrogated and given two choices: to confess (denoted by D from "Defection") or to remain silent (denoted by C from "Cooperation"). Depending on their choices their utilities from their prison sentences look as follows:

	D	C
D	2 , 2	7 , 0
C	0 , 7	5 , 5

Which are the NE of this game?

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Example (Second-price auction)

Ann and Bob are risk-neutral and bid for a bike which is auctioned according to a second-price auction. They can only bid an amount from $\{10, 20, 30, 40\}$. If they submit the same bid, we toss a coin. Ann's valuation of the bike is 20 whereas Bob's is 30.

	10	20	30	40
10	5,10	0,20	0,20	0,20
20	10,0	0,5	0,10	0,10
30	10,0	0,0	-5,0	0,0
40	10,0	0,0	-10,0	-10,-5

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Which are the NE of this game?

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There may exist no NE.

Example (Matching pennies)

Ann and Bob are given a coin each and are asked to place their coins either Heads up (H) or Tails up (T). If they choose the same side of the coin then Ann gets both coins, whereas if they choose opposing sides then Bob gets them both. Their utilities look as follows:

	H	T
H	1 , -1	-1 , 1
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Which are the NE of this game?

- What is the reason there are no NE in this game?

	H	T
H	1 , -1	-1 , 1
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- What is a potential solution?

- What is the reason there are no NE in this game?

	H	T
H	1 , -1	-1 , 1
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- What is a potential solution?

Become unpredictable!

- A player can become unpredicable by randomizing

Definition

A **mixed strategy** of player i is a lottery p^i over the actions A^i.

- Each (pure) strategy a^i will occur with probability $p^i(a^i)$
- So, we have two types of probability:
  1. Belief about opponent's strategies (subjective probability)
  2. Randomization over own strategies (objective uncertainty)

- A **mixed Nash Equilibrium (NE)** reflects the same idea

  A mixed strategy profile from which no player wants to unilaterally deviate

- Essentially we still assume the following:
  1. **Behavior:** All players choose optimally given their beliefs
  2. **Strategic reasoning:** All players have correct beliefs
- The difference is that strategies/beliefs may involve probabilities.
- Since beliefs are correct, in equilibrium a player's belief about another player coincides with this other player's mixed strategy
- That's why we do not refer explicitly to beliefs

Definition

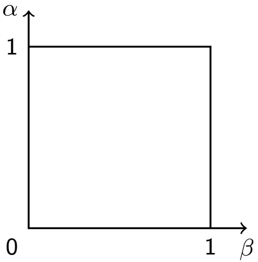
A mixed strategy profile (p^1, p^2) is a mixed NE if for every $i \in N$ and every a^i that receives positive probability by p^i:

$$U^i(a^i, p^{-i}) \geq U^i(x^i, p^{-i}) \text{ for every } x^i \in A^i$$

Theorem (Nash, 1951)

Every game has a mixed NE.

		(β) H	($1-\beta$) T
(α)	H	1 , -1	-1 , 1
($1-\alpha$)	T	-1 , 1	1 , -1



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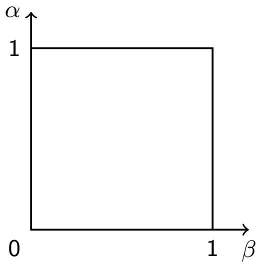

The best responses of Ann (blue):

$$
\begin{aligned}
\alpha = 1 \quad &\Leftrightarrow \quad H \succ^a T \\
&\Leftrightarrow \quad \beta - (1-\beta) > -\beta + (1-\beta) \Leftrightarrow \beta > 1/2 \\
\alpha = 0 \quad &\Leftrightarrow \quad H \prec^a T \\
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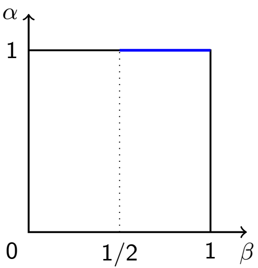

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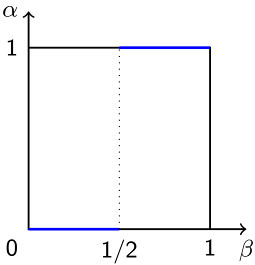

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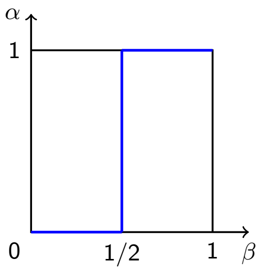

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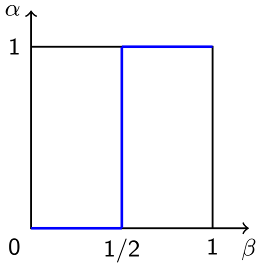

The best responses of Bob (red):

$$
\begin{aligned}
\beta = 1 \quad &\Leftrightarrow \quad H \succ^b T \\
&\Leftrightarrow \quad -\alpha + (1-\alpha) > \alpha - (1-\alpha) \Leftrightarrow \alpha < 1/2 \\
\beta = 0 \quad &\Leftrightarrow \quad H \prec^b T \\
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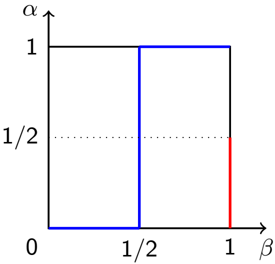

The best responses of Bob (red):

$$
\begin{aligned}
\beta = 1 \quad &\Leftrightarrow \quad H \succ^b T \\
&\Leftrightarrow \quad -\alpha + (1-\alpha) > \alpha - (1-\alpha) \Leftrightarrow \alpha < 1/2 \\
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0 < \beta < 1 \quad &\Leftrightarrow \quad H \sim^b T \\
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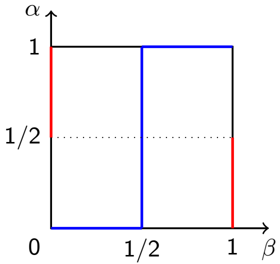

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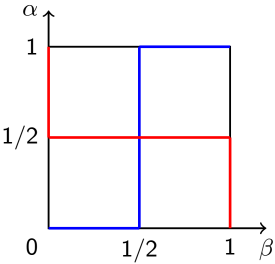

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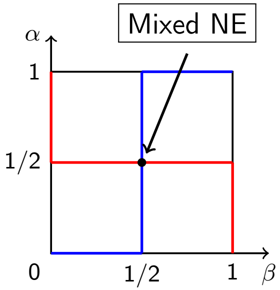

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Definition

A two-player game is **zero-sum** if one's gain is the other's loss:

$$u^1(a, b) = -u^2(a, b)$$

Theorem (Maxmin Theorem (vNM, 1944))

In a two-player zero-sum game, (p^a, p^b) is a NE

if and only if

for each player i and for every mixed strategy q^i:

$$u^i(p^i, p^{-i}) = \max_{q^i} \min_{q^{-i}} u^i(q^i, q^{-i})$$

- Each player maximizes his minimum expected utility:

  You optimize your worst-case scenario.

- Such strategy is used if a player is very conservative:

  It focuses on the opponent not being able to exploit you.

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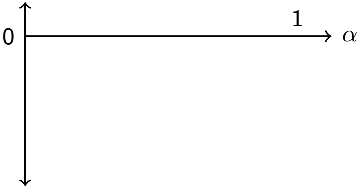

- Ann's expected utility is equal to:

$$U^a(\alpha, \beta) = \beta(2\alpha - 1) + (1 - \beta)(1 - 2\alpha)$$

- For each α, find Bob's strategy β that makes Ann worse off:

$$\begin{aligned}
\alpha \leq 1/2 \quad & \Rightarrow \quad 2\alpha - 1 \leq 1 - 2\alpha \\
& \Rightarrow \quad U^a(\alpha, \beta) \text{ is minimized when } \beta = 1 \\
& \Rightarrow \quad \min_{\beta} U^a(\alpha, \beta) = 2\alpha - 1 \\
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& \Rightarrow \quad U^a(\alpha, \beta) \text{ is minimized when } \beta = 0 \\
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\end{aligned}$$

- So $\min_{\beta} U^a(\alpha, \beta)$ is maximized at $\alpha = 1/2$. Intuition?

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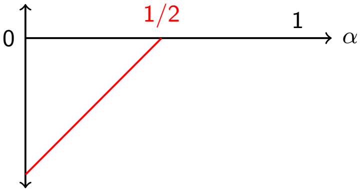

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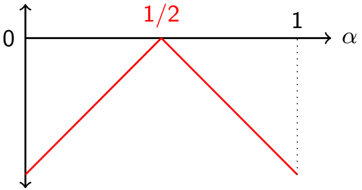

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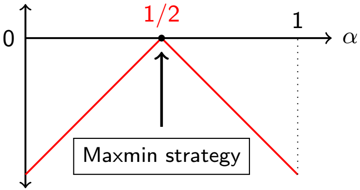

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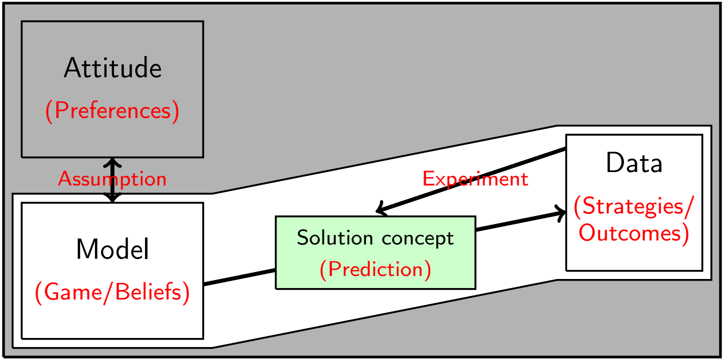
How do we make sense of the solution concept?



- In zero-sum games, players are conservative and choose safe
- In general, players correctly guess what the opponent does:
    1. Convention or norm
    2. Unmodelled learning process
    3. Conscious reasoning, e.g., unique profile surviving ISD

- At home you need to read Chapters 15.1,15.2, 15.3, 15.5, 15.7, 15.8 of the textbook Edition 2020).
- As of the rest, the same instructions as last week apply